GENERALIZATIONS OF THE NOTION OF CLASS GROUP[1,2]

BY

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Introduction

There are currently available two equivalent descriptions for the class group of a Noetherian integrally closed domain. The older, more direct approach, can be summarized as follows: Let A be a Noetherian integrally closed domain and let D denote the free abelian group with the prime ideals of A of height one as generators. Let $x \neq 0$ be an element of A and consider the element $\sum_{\mathrm{bt}\,\mathfrak{p}=1} l_{A_\mathfrak{p}}(A_\mathfrak{p}/xA_\mathfrak{p})\cdot\mathfrak{p}$ of D. Let R denote the subgroup of D generated by all such elements. Then the class group of A, $C(A)$, is the group D/R.

The second approach will now be described. Let A be a Noetherian integrally closed domain. Let $\mathfrak{M}_i$ denote the category of all finitely generated A-modules M such that $M_\mathfrak{p} = 0$ for all prime ideals of height less than i. In other words, $\mathfrak{p} \in \operatorname{Supp} M$ if and only if the height of $\mathfrak{p}$ is at least i. From the exact sequence of categories

$$0 \to \mathfrak{M}_1/\mathfrak{M}_2 \to \mathfrak{M}_0/\mathfrak{M}_2 \to \mathfrak{M}_0/\mathfrak{M}_1 \to 0$$

derives an exact sequence of Grothendieck groups

$$K^0(\mathfrak{M}_1/\mathfrak{M}_2) \to K^0(\mathfrak{M}_0/\mathfrak{M}_2) \to K^0(\mathfrak{M}_0/\mathfrak{M}_1) \to 0.$$

Now $K^0(\mathfrak{M}_0/\mathfrak{M}_1)$ is $\mathbf{Z}$; the isomorphism is given by

$$M \to \dim_F(F \otimes_A M)$$

where F is the field of quotients of A. Therefore

$$K^0(\mathfrak{M}_0/\mathfrak{M}_2) \cong \mathbf{Z} \oplus \operatorname{Im}(K^0(\mathfrak{M}_1/\mathfrak{M}_2)).$$

$\operatorname{Im}(K^0(\mathfrak{M}_1/\mathfrak{M}_2))$ can be identified as the class group, $C(A)$, of A [2, Chap. 7, § 4, n° 7, Prop. 17].

In this article we generalize both these definitions to prime ideals of height greater than 1. Generalizing from the first description a sequence of groups, to be called $C_i(A)$ $(0 \leq i \leq \dim A)$, is obtained; from the second description a sequence of groups, to be called $W_i(A)$ $(0 \leq i \leq \dim A)$, is obtained.

The groups $W_i(A)$ are defined for each commutative Noetherian ring A. The groups $C_i(A)$ are defined for those commutative Noetherian rings A which are locally Macaulay.

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[2] This article relates the completion and extension of work of which [4] was the research announcement.

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