A MINIMAL REPRESENTATION FOR THE LIE ALGEBRA $\mathfrak{E}_7$

BY
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The smallest representation of a split Lie algebra of type E_7 over a field of characteristic zero has dimension fifty-six and is closely related to the split exceptional simple Jordan algebra. This representation has been studied by Freudenthal [4] and Seligman [11]. We show that it is possible to define a multiplication and a trace on the representation space so that the Lie algebra $\mathfrak{E}_7$ is realized by the derivations and left multiplications by elements of trace zero. The derivations alone form a Lie algebra of type E_6. The Killing forms of these Lie algebras are also presented. Later we slightly generalize the fifty-six dimensional algebras so as to obtain as algebras of derivations a class of Lie algebras of type E_6 including the "twisted" algebras. Although these include all real forms of $\mathfrak{E}_6$, no new forms of $\mathfrak{E}_7$ occur. Finally a method of twisting E_7's is given. This results in a class which contains all real forms of $\mathfrak{E}_7$.

Throughout, the characteristic of the ground field is not two or three.

1. Cayley and Jordan algebras

In this section we collect some facts about Cayley algebras and exceptional simple Jordan algebras. The following properties of Cayley algebras are proved in [6]. A Cayley algebra $\mathfrak{C}$ is an eight-dimensional central simple alternative algebra. It possesses an involution $x \to x^*$ such that the quadratic norm form $n(x) = xx^*$ permits composition: $n(xy) = n(x)n(y)$. We linearize $n(x)$ to obtain a non-degenerate symmetric bilinear form on $\mathfrak{C}$:

$$(x, y) = \tfrac{1}{2}[n(x + y) - n(x) - n(y)] = \tfrac{1}{2}(xy^* + yx^*).$$

A Cayley algebra has an identity 1 and every element is of the form $\alpha 1 + x_0$ with $(x_0, 1) = 0$. Then $(\alpha 1 + x_0)^* = \alpha 1 - x_0$. A basis can be chosen for $\mathfrak{C}$ so that the norm form becomes

$$n(x) = \xi_0^2 - \rho\xi_1^2 - \sigma\xi_2^2 + \rho\sigma\xi_3^2 - \tau(\xi_4^2 - \rho\xi_5^2 - \sigma\xi_6^2 + \rho\sigma\xi_7^2).$$

Here ρ, σ, τ are non-zero field elements. To exhibit them we will write $\mathfrak{C} = \mathfrak{C}(\rho, \sigma, \tau)$, even though they are not uniquely determined by $\mathfrak{C}$. A Cayley algebra with zero divisors is called split.

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