RATIONAL FUNCTIONS H^∞ AND H^p ON INFINITELY CONNECTED DOMAINS

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Introduction

Let D be a domain (an open connected set) in the complex plane and for $1 \leq p < \infty$, let $H^p(D)$ be those analytic functions f on D for which $|f|^p$ has a harmonic majorant on D. Fix t_0 in D and put $\| f \|_p = [u(t_0)]^{1/p}$ where u is the least harmonic majorant of $|f|^p$ on D. Then $\| \quad \|_p$ is a norm on $H^p(D)$ which depends upon the point t_0 although the resulting topology on $H^p(D)$ does not. Let $H^\infty(D)$ be the algebra of bounded analytic functions on D with the uniform norm.

These H^p spaces, which generalize the classical Hardy H^p spaces in the unit disc U for $1 \leq p < \infty$, were introduced by Parreau in 1951 [5] and independently by Rudin in 1955 [6]. In his paper Rudin showed that if D is bounded by a finite number of disjoint circles then the rational functions with poles off $\bar{D}$ are dense in $H^p(D)$, $1 \leq p < \infty$, and hence $H^\infty(D)$ is dense in $H^p(D)$ for $1 \leq p < \infty$. Further, if D_1 is conformally equivalent to D, then $H^p(D_1)$ and $H^p(D)$ are isometrically isomorphic for $1 \leq p \leq \infty$; hence $H^\infty(D)$ is dense in $H^p(D)$ on all bounded domains with only finitely many complementary components. The aim of this paper is to show that $H^\infty(D)$ is dense in $H^p(D)$ on two types of infinitely-connected domains. These two types of domain are very different and the techniques of the proof differ vastly from one to the other. One type is treated in Section 1 and the other in Section 2. The author would like to thank Profs. F. Forelli and M. Voichick for several helpful conversations regarding the contents of this paper.

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If $H^\infty(D)$ contains non-trivial functions, then the unit disc U is the universal covering surface of D and hence there is an analytic function w from U onto D which is locally one-to-one and may be used to lift paths uniquely from D to U. If $f \in H^p(D)$, $1 \leq p \leq \infty$, then the analytic function $g(z) = f(w(z))$ is in $H^p(U)$ and if $w(0) = t_0$ (which we may assume without loss of generality), then $\| g \|_p = \| f \|_p$, where

$$\| g \|_p^p = \sup_{0<r<1} \left[\frac{1}{2\pi} \int_0^{2\pi} | g(re^{i\theta}) |^p \, d\theta \right] \qquad \text{for } 1 \leq p < \infty$$

and $\| g \|_\infty = \sup_{z \in U} | g(z) |$ is the usual H^p norm in the disc.

Let G be the group of linear fractional transformations T of U onto U such that $w(T(z)) = w(z)$ for all z in U. If $f \in H^p(D)$ and if $g = f \circ w$, then g is

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