TOPOLOGICALLY UNKNOTTING TUBES IN EUCLIDEAN SPACE

BY
R. C. LACHER[1]

In this paper we consider closed, locally flat embedding of tubes $B^{k-1} \times R^1$ and $S^{k-1} \times R^1$ into R^n. In Part I we show that $B^{k-1} \times R^1$ knots in R^3 but unknots in R^n if $n \geq 4$. The situation with $S^{k-1} \times R^1$ is more complicated. In Parts II and III, we show that $S^{k-1} \times R^1$ can knot in R^{k+2} and in R^{2k} and in most R^n for $k + 2 \leq n \leq 2k$. Thus a general low-codimensional unknotting theorem is nonexistent. However, in Part IV we show that any closed, locally flat embedding of $S^{k-1} \times R^1$ in R^n, $k \leq n - 3$, is unknotted provided that it is "unlinked at infinity", a condition derived while proving that the examples in Part III actually knot. A corollary is that $S^{k-1} \times R^1$ unknots in R^n if $n \geq 2k + 1$, $k \geq 2$.

Embeddings of $S^{n-2} \times R^1$ into R^n are studied in Part V.

Several discussions with Joe Martin were helpful in the formulation of Parts II and III.

Added in Proof. Closed, locally flat embeddings of $S^{k-1} \times R^1$ in R^n are classified by the homotopy group $\pi_{k-1}(S^{n-k-1})$, provided $3(k + 1) < 2n$.

Definitions and Notation. We think of B^n as the closed unit ball in euclidean n-space R^n, and we identify R^k with $R^k \times 0$ in R^n. Also, S^n is the boundary of B^{n+1}. Thus $B^k \times R^{n-k} \subset R^n$ and $S^{k-1} \times R^{n-k} \subset R^n$. $\hat{R}^n$ is used to denote the one-point compactification of R^n. Of course, $\hat{R}^n$ is homeomorphic to S^n.

Let K be a (topological) k-manifold contained in the interior of the n-manifold N. K is *locally flat at the point* $x \in \operatorname{Int} K$ (the *interior* of K) if x has a neighborhood U in N such that $(U, U \cap K)$ and (R^n, R^k) are homeomorphic as pairs. K is *locally flat at the point* $x \in \operatorname{Bd} K$ (the *boundary* of K) if x has a neighborhood U in N such that $(U, U \cap K)$ and (R^n, R^k_+) are homeomorphic as pairs, where $R^k_+ = R^{k-1} \times [0, \infty) \subset R^k$.

An embedding f of a k-manifold K into the interior of the n-manifold N is *locally flat at the point* $x \in K$ if $f(K)$ is locally flat at x; f is called a *locally flat embedding* if f is locally flat at every point of K.

Finally, an embedding is *closed* if its image is a closed subset of its range.

Part I. Unknotting $B^{k-1} \times R^1$ in R^n for $n \geq 4$

Before stating the main unknotting theorem, we prove two propositions. The first says essentially that "setwise" unknotting implies "pointwise" unknotting. The second shows that knotting occurs in dimension three.

Received March 28, 1967.

[1] Partially supported by a National Science Foundation grant.