CRITERIA FOR POSITIVE GREEN'S FUNCTIONS

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If L is an elliptic operator defined by

$$(1) \qquad Lv = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial v}{\partial x_i}\right) - 2\sum_{i=1}^{n} b_i \frac{\partial v}{\partial x_i} + cv$$

and $c(x) \geq 0$ in a bounded domain $D \subset R^n$, then the Green's function $G(x, \xi)$ associated with L on D is known to be positive in $D \times D$ whenever it exists. This fact follows readily from the characteristic properties of $G(x, \xi)$ and the Hopf maximum principle which applies to solutions of $Lv = 0$ when $c \geq 0$.

In this paper we shall use recently proved comparison theorems for elliptic equations to establish more general conditions under which fundamental solutions of (1) are non-negative in $D \times D$. Such conditions will be seen to involve *all* the coefficients of L and, in the case L is self-adjoint, will reduce to the assumption that the smallest eigenvalue of

$$(2) \qquad \begin{aligned} Lu &= \lambda u \quad \text{in } D \\ u &= 0 \quad \text{on } \partial D \end{aligned}$$

be positive.

Comparison theorems for elliptic equations deal with solutions v of $Lv = 0$ and u of $\Lambda u = 0$ where L is given by (1) and

$$(3) \qquad \Lambda u = -\sum \frac{\partial}{\partial x_j}\left(\alpha_{ij}\frac{\partial u}{\partial x_i}\right) - 2\sum_i \beta_i \frac{\partial u}{\partial x_i} + \gamma u.$$

If there exists a non-trivial solution u of $\Lambda u = 0$ in a domain $D_0 \subset D$ which vanishes on ∂D_0, and if the operator L is "smaller" than Λ in an appropriate sense to be made precise below, then such comparison theorems assert that every solution of $Lv = 0$ has a zero in $\bar{D}_0$.

For non self-adjoint equations of the form $Lv = 0$ where, L is given by (1), we make use of a comparison theorem due to Swanson [1].

In order to make the matrix

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} & -b_1 \\ \vdots & & & \vdots \\ a_{n1} & \cdots & a_{nn} & -b_n \\ -b_1 & \cdots & -b_n & g \end{pmatrix}$$

positive semidefinite, Swanson formulates the condition

$$(4) \qquad g \det(a_{ij}) \geq -\textstyle\sum_{i=1}^{n} b_i B_i,$$

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