DECOMPOSITION OF PURE SUBGROUPS OF TORSION FREE GROUPS

BY

PHILLIP A. GRIFFITH[1]

1. Introduction

Throughout this paper all groups are abelian. The notion of a cotorsion group, introduced by Harrison in [8], plays an important role. Some basic properties of cotorsion groups are listed in [4]. A torsion free group is called completely decomposable if it is isomorphic to a direct sum of torsion free groups of rank one. If G is a torsion free group and H is a subgroup of G, we use the symbol H_* to denote the minimal pure subgroup of G containing H. The symbols $\sum$ and $+$ will be used for direct sums; whereas the subgroup of a group G generated by subsets S and T will be denoted by $\{S, T\}$.

Recently, the author gave a negative answer [7] to a question posed by E. Weinberg [9] which asked: Does there exist a torsion free abelian group of cardinality greater than the continuum with the property that each pure subgroup is indecomposable? In this paper we use the techniques of [7] to generalize our result concerning Weinberg's question. In fact, if G is a torsion free group we show that there is a completely decomposable pure subgroup C of G such that $|G| \leq |C|^{\aleph_0}$. Our investigation of completely decomposable pure subgroups of torsion free groups requires the study of a distinguished class of independent subsets of a torsion free group. An independent subset S of a torsion free group G will be called quasi-pure independent if $\sum_{x \in S} \{x\}_*$ is a pure subgroup of G and $\{x\}_* = \{x\}$ whenever $\{x\}_*$ is cyclic and $x \in S$. Note that $\{S\}_* = \sum_{x \in S} \{x\}_*$ if S is a quasi-pure independent. We remark that quasi-pure independence is equivalent to pure independence if G is $\aleph_1$-free. In Section 2 we establish a number of remarkable properties of quasi-pure independent subsets.

2. Quasi-pure independence

We observe that, although nonvoid pure independent subsets may not exist, nonzero torsion free groups always have quasi-pure independent subsets. The proof of the following proposition can be accomplished by standard techniques.

PROPOSITION 2.1. *Any quasi-pure independent subset S of a torsion free group G is contained in a maximal quasi-pure independent subset of G.*

One might hope that the cardinality of a maximal quasi-pure independent subset of a torsion free group is an invariant of the group. In [6] it was shown

Received March 6, 1967.

[1] The author wishes to acknowledge support by the National Aeronautics and Space Administration.