SOME EXAMPLES FOR WEAK CATEGORY AND CONILPOTENCY

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1. Introduction

We are concerned here with certain numerical invariants of homotopy type akin to the Lusternik-Schnirelmann category.

It is known that cat B, the Lusternik-Schnirelmann category of a space B (when renormalized) is an upper bound for conil B, the conilpotency class of the suspension of B [18; Theorem 2.10]. Furthermore if B is an $(n - 1)$-connected CW-complex of dimension $\leq (k + 2)n - 2$ and conil $B \leq k$ then cat B = conil B [2; Theorem 2].

Berstein and Hilton [3; (2.1)] gave a definition of category which is equivalent, for most classes of spaces, to the original one of Lusternik and Schnirelmann. This definition suggests two other invariants, wcat B, the weak category of a space B and wcat e, the weak category of the natural embedding map $e : B \to \Omega\Sigma B$ [3; (2.2)], [7; §5]. These two weak categories take values lying between those of cat B and conil B, but we will show by examples in Section 2 that all the invariants are different.

None of these definitions of category and weak category dualize easily in the sense of Eckmann-Hilton. So Ganea introduced yet another definition of category and weak category, in terms of a 'ladder' of fibrations, which does dualize. We will denote these invariants by G-cat and G-wcat respectively. (See Definition 6.1 of [5] for the cocategory of a space.) In Sections 3 and 4 we will show that G-cat B is the same invariant as cat B but that G-wcat B is different from wcat B.

We collect together the results on the relationships between the various invariants in the following theorem. All the numerical invariants in this paper will be normalized so as to take the value 0 on contractible spaces.

THEOREM 1.1. *Let B have the homotopy type of a simply connected countable* CW*-complex; then*

$$\text{cat } B = \text{G-cat } B \geq \text{G-wcat } B \geq \text{wcat } B \geq \text{wcat } e \geq \text{conil } B \geq \cup\text{-long } B$$

and furthermore all the inequalities can occur.

Here $\cup$-long B is the length of the longest nontrivial cup product of positive dimensional elements of $H^*(B; R)$, where R is any commutative ring.

Theorem 1.1 will follow from Theorems 3.4 and 3.5, [7; Theorems 4.4 and 5.2] and the remaining two inequalities follow directly from the definitions.

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