ON SEMIGROUPS GENERATED BY TOPOLOGICALLY NILPOTENT ELEMENTS

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In a Banach algebra A one can define, for each element x,

$$\operatorname{ixp} x = -\sum_{n=1}^{\infty} x^n/n!$$

or, if A has identity e, $\exp x = e - \operatorname{ixp} x$. Then, for a given x, $\alpha \to \operatorname{ixp} \alpha x$ ($\alpha \to \exp \alpha x$) is a one-parameter semigroup under circle composition (group under multiplication). In this paper we deal with the following question:

How "bounded" can such a semigroup be when the generator x is "topologically very nilpotent"?

In [4, pp. 248, 259] the following results are given:

I. If $\lim_{n\to\infty} \| x^n \|^{1/n} = 0$ and $x \neq 0$, then $\operatorname{ixp} \alpha x$ can not be bounded in norm for all α.

II. If $\lim n \| x^n \|^{1/n} = 0$ and $x \neq 0$, then $\operatorname{ixp} \alpha x$ can not be bounded in norm for $\alpha > 0$.

This type of problem has arisen in two different contexts. In the paper by Bohnenblust and Karlin [1] on the geometry of the unit sphere in Banach algebras they make a conjecture, which can be shown to be equivalent to

"assumption of I $\Rightarrow$ conclusion of II".

This is not true, as has been shown in [7], but apparently I and II are the affirmative results corresponding to that conjecture. On the other hand, I is a key result in the author's classification of real Banach algebras [4].

In improving these results in several different directions, we use methods from the theory of functions, in particular conditions for perfectly regular growth of an entire function. The sharpest result in the relevant direction seems to a recent theorem by Essén [2], which we quote here in the form most suitable for the application.

Let f be an entire function with $f(0) = 1$, maximum modulus M and minimum modulus m. ˥or a number λ, $0 < \lambda < 1$, let

$$K_\lambda(R) = R^{-\lambda} \log M(R)$$

$$I_\lambda(R) = \int_0^R r^{-1-\lambda} [\log m(r) - \cos \pi\lambda \log M(r)]\, dr$$

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