MAPPING CUBES WITH HOLES ONTO CUBES WITH HANDLES

BY

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1. Introduction

In connection with some work by W. Haken [4] on the Poincaré conjecture in dimension 3, R. H. Bing raised the following question in [2]. If K_2 is any cube with 2 holes, does there always exist a continuous map f of K_2 onto a cube with 2 handles C_2 such that $f \mid \text{Bd } K_2$ is a homeomorphism onto Bd C_2? (We call such a map f a boundary preserving map of K_2 onto C_2.) In general, if K_n is a cube with n holes, does there always exist a boundary preserving map of K_n onto a cube with n handles C_n? For the case $n = 1$, J. Hempel in Theorem 5 of [5] answered the question in the affirmative. In Theorem 1 of this paper we show that the question has a negative answer for $n = 2$. It then follows, as a corollary to Theorem 1, that the question has a negative answer for $n \geq 2$. Theorem 2 gives a necessary and sufficient condition for the existence of a boundary preserving map of K_n onto C_n. Theorem 3 gives another sufficient condition for the existence of a boundary preserving map of K_2 onto C_2.

2. Terminology

Throughout this paper all sets which appear can be considered as polyhedral subsets of E^3. *A cube with n holes* K_n and *a cube with n handles* C_n are defined as on pages 90 and 95 of [2]. Any cube with holes or handles is to be thought of as a polyhedral subset of E^3. In analogy to the definition of 1-linked simple closed curves (scc's) in E^3 [9], we define disjoint scc's X, Y to be *1-linked in the 3-manifold M* if for each pair of compact orientable 2-manifolds M_X and M_Y in M such that Bd $M_X = X$ and Bd $M_Y = Y$, it follows that $M_X \cap M_Y \neq \emptyset$. At the end of Section 4 we note an analogy between the main result of this paper and the example of a boundary link $l_1 \cup l_2$ given in [9].

Suppose g is a map of K_n onto C_n. Then g is said to be a boundary preserving map of K_n onto C_n if g is continuous and $g \mid \text{Bd } K_n$ is a homeomorphism onto Bd C_n. It can be shown that if g is a boundary preserving map of K_n onto C_n, then there is a piecewise linear map f of K_n onto C_n and a product neighborhood θ_1 ($=$ Bd $K_n \times [0, 1]$) of Bd K_n in K_n and a product neighborhood θ_2 of Bd C_n in C_n such that (1) $f \mid \theta_1$ is a homeomorphism onto θ_2 and (2) $f(K_n - \theta_1) = C_n - \theta_2$. We will assume then that any boundary preserving map f of K_n onto C_n has been adjusted so that it is piecewise linear and satisfies (1) and (2) above.

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