APPROXIMATION TO HARMONIC FUNCTIONS

BY

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Let K be a compact plane set. In what follows we use, for the most part, the notation and terminology of [4]. We let $D(K)$ be the set of functions continuous on K and harmonic on the interior of K, and let $H(K)$ be the set of functions harmonic in a neighborhood of K. $\overline{H(K)}$ will denote the uniform closure of $H(K)$ on K. It is known that the Choquet boundary of $D(K)$ is just the set R of regular points of K. We let P denote the Choquet boundary of $\overline{H(K)}$. It is known that $D(K) = \overline{H(K)}$ if and only if the capacity of $\partial K - P$ is zero. In a given case this condition may not be so easy to check since it requires some knowledge of the set P. In this note we give a sufficient condition that $D(K) = \overline{H(K)}$, which seems more "geometrical" in nature.

For each $p \in K$ there is a unique positive measure $d\mu_p$ carried on R such that $u(p) = \int u d\mu_p$ for all $u \in D(K)$. This is called the harmonic measure for p. There is also a unique positive measure $d\nu_p$ carried on P such that $u(p) = \int u d\nu_p$ for all $u \in \overline{H(K)}$. This measure is called the Keldysh measure for p. We let $\partial_0 K$ be the set of points in ∂K that lie on the boundary of some component of the complement of K; and let $\partial_i K$ be the set of points in ∂K that lie on the boundary of some component of the interior of K. In what follows dm will denote planar Lebesgue measure. We will show:

Theorem. *If*

$$d\mu_p(\partial K - \partial_0 K) = 0 \quad \textit{for all} \quad p \in \operatorname{int} K \tag{1}$$

$$dm(\partial K - (\partial_0 K \cup \partial_i K)) = 0, \tag{2}$$

then $\overline{H(K)} = D(K)$.

(Note that (2) holds if int K is dense in K and has finitely many components.)

We will give examples to show that neither (1) nor (2) alone is sufficient to give the conclusion.

The method of proof is that used by Carleson in his proof of Walsh's Theorem [2]. See also [3].

Let α be a real measure carried on ∂K. We let

$$\hat{\alpha}(\xi) = \int \log \frac{1}{|z - \xi|} \, d\alpha(z).$$

We say $\hat{\alpha}(\xi_0)$ exists if

$$\int \log \frac{1}{|z - \xi_0|} \, d|\alpha|(z) < \infty.$$

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