THE NUMBER OF HALL π-SUBGROUPS OF A FINITE GROUP

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This note gives a theorem on the number of Hall π-subgroups of a finite group which includes a recent result of Marshall Hall on the number of Sylow subgroups as well as the classical theorem of Philip Hall on the number of Hall π-subgroups of a solvable group (cf. [1] and [2]).

We shall consider groups G which satisfy the following proposition for a given set of primes π.

A_π. Given any π-subgroups P_i of G for $i = 1, 2$, there are Hall π-subgroups H_i of G so that $H_i \geq P_i$ and an automorphism α of G so that $H_1 \alpha = H_2$.

If a group satisfies proposition A_π we shall call it an A_π-group. It is clear that a group satisfying the well-known proposition D_π of Philip Hall (cf. [3]) is an A_π-group. But the class of A_π-groups is larger than the class of groups satisfying D_π since for instance the projective group PSL(2, 7) of order 168 has two classes of subgroups isomorphic to the symmetric group S_4 which are conjugate in the automorphism group of PSL(2, 7). It is also clear that an A_π-group satisfies proposition E_π of [3] and that there are E_π-groups not A_π-groups; for instance, PSL(2, 11) of order 660 which has two non-isomorphic groups of order 12.

Before stating the main theorem it will be convenient to have the following lemma whose easy proof is omitted.

LEMMA. *Let the group D be the direct product of groups G_i for $i = 1, 2, \cdots, n$, where each G_i is isomorphic to a given group G. Then a Hall π-subgroup of D is the product of Hall π-subgroups H_i of G_i and D is an A_π-group if and only if G is.*

The main theorem is as follows.

THEOREM. *Let G be a finite A_π-group for a certain set of primes π: then the number $n_\pi(G)$ of Hall π-subgroups of G is a product of integers such that each integer is either the number of Hall π-subgroups of a simple A_π-group or is a prime power congruent to 1 modulo a prime of π.*

The proof is by induction on $|G|$ the order of G. If G is a direct product of isomorphic simple groups the theorem follows from the above lemma. Accordingly we consider the case where G has a proper non-trivial characteristic subgroup K, which we shall assume to be minimal. We let $\bar{G}$ denote G/K.

Case I. K is a π-group. It is easy to see then that there is a one-one cor-

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