INTEGRAL REPRESENTATIONS OF FRACTIONAL POWERS OF INFINITESIMAL GENERATORS

BY

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Introduction

The main purpose of this paper is to give a class of integral representations for the fractional powers $(-A)^\alpha$, where $0 < \alpha$ and A is the infinitesimal generator of a bounded strongly continuous semigroup T_t of bounded linear operators on a Banach space X. The definition of $(-A)^\alpha$ used in [3] is

$$\lim_{\varepsilon\to 0} C \int_\varepsilon^\infty t^{-\alpha-1}(I - T_t)^r f\, dt \tag{0.1}$$

where $0 < \alpha < r$, r is a positive integer and C is an appropriate constant. For the case $0 < \alpha < 1$, $r = 1$, the above definition of $(-A)^\alpha$ can be motivated by noting that if $a < 0$, then by a simple change of variable

$$\int_0^\infty t^{-\alpha-1}(1 - e^{ta})\, dt = (-a)^\alpha \int_0^\infty t^{-\alpha-1}(1 - e^{-t})\, dt;$$

so $(-a)^\alpha$ is a constant times the integral on the left. Komatzu [2] has shown that the operator defined by (0.1) can also be represented in the form

$$\lim_{\varepsilon\to 0} C \int_\varepsilon^\infty t^{-\alpha-1}(-tA(I - tA)^{-1})^r f\, dt. \tag{0.2}$$

A similar motivation could be given for this integral representation.

In this paper we introduce "kernels"

$$S(\sigma_{(t)}) = \int_0^\infty T_u\, d\sigma_{(t)}(u)$$

where $d\sigma_{(t)}(u) = d\sigma(u/t)$ and show (see Theorem 1.4) that limits of the form

$$\lim_{\varepsilon\to 0} \int_\varepsilon^\infty t^{-\alpha-1} S(\sigma_{(t)}) f\, dt$$

all define the same operator as σ ranges over a wide class of measures. In Section 2 we show that the "kernels" in (0.1) and (0.2) correspond to special choices of σ within the class. In Section 3 we show that the class cannot be enlarged by establishing a converse to Theorem 1.4 (see Theorem 3.1). In Section 4 we define Lipschitz spaces corresponding to the "kernels" $S(\sigma_{(t)})$ and prove a Lions-Peetre type theorem, Theorem 4.1, relating these Lipschitz spaces to

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