HOMOTOPY GROUPS OF PRO-SPACES

BY

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1. Introduction

In this paper we continue the investigation [4], [5] of the homotopy theory of pro-spaces indexed over the positive integers. It is known that the homotopy type of a "nice" pro-space $\{X_s\}$ is dependent upon (among other things) its homotopy pro-groups $\{\pi_n X_s\}$. We show here that in fact, homotopy groups $\pi_n \{X_s\}$ —defined as the set of homotopy classes of maps from a kind of pro-*n*-sphere $\{S_s^n\}$ into $\{X_s\}$ —capture the same information as $\{\pi_n X_s\}$. More generally we show that pro-groups indexed over the positive integers contain no more information than groups, by exhibiting a functor *P* from such pro-groups to groups, such that a map *f* between pro-groups is an isomorphism if and only if *Pf* is an isomorphism.

In Section 2 we review pro-spaces and define the homotopy groups. The more general algebraic situation is discussed in Section 3. In Section 4 we show that $\pi_n\{X_s\} \cong P\{\pi_nX_s\}$ and comment on the connection with the proper homotopy groups of a complex.

2. Pro-spaces

For more details see [4]. Let \mathscr{S}_0 be the category of pointed, connected spaces, i.e., pointed, connected simplicial sets; * is the basepoint or a one-point space. Then tow- \mathscr{S}_0 consists of towers in \mathscr{S}_0 ,

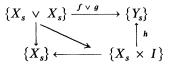
$$\cdots \to X_{s+1} \to X_s \to \cdots \to X_1 \to X_0 = *,$$

denoted $\{X_s\}$, and informally called a pro-space, with maps defined by

$$\operatorname{Hom}_{\operatorname{tow-}\mathscr{G}_{0}}\left(\{X_{s}\}, \{Y_{s}\}\right) = \lim_{\stackrel{\leftarrow}{j}} \lim_{i} \operatorname{Hom}_{\mathscr{G}_{0}}\left(X_{i}, Y_{j}\right)$$

Similar definitions apply to tow- \mathscr{G} and tow- \mathscr{A} where \mathscr{G} is the category of groups, and \mathscr{A} is the category of abelian groups.

For $n \ge 1$, the *n*th homotopy pro-group of $\{X_s\}$ is the pro-group $\{\pi_n X_s\}$. We say that two maps, f and g, from $\{X_s\}$ to $\{Y_s\}$ are homotopic if there is a map $h: \{X_s \times I\} \to \{Y_s\}$ such that the diagram



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