HOMOTOPY GROUPS OF PRO-SPACES

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1. Introduction

In this paper we continue the investigation [4], [5] of the homotopy theory of pro-spaces indexed over the positive integers. It is known that the homotopy type of a "nice" pro-space $\{X_s\}$ is dependent upon (among other things) its homotopy pro-groups $\{\pi_n X_s\}$. We show here that in fact, homotopy groups $\pi_n\{X_s\}$—defined as the set of homotopy classes of maps from a kind of pro-n-sphere $\{S_s^n\}$ into $\{X_s\}$—capture the same information as $\{\pi_n X_s\}$. More generally we show that pro-groups indexed over the positive integers contain no more information than groups, by exhibiting a functor P from such pro-groups to groups, such that a map f between pro-groups is an isomorphism if and only if Pf is an isomorphism.

In Section 2 we review pro-spaces and define the homotopy groups. The more general algebraic situation is discussed in Section 3. In Section 4 we show that $\pi_n\{X_s\} \cong P\{\pi_n X_s\}$ and comment on the connection with the proper homotopy groups of a complex.

2. Pro-spaces

For more details see [4]. Let $\mathscr{S}_0$ be the category of pointed, connected spaces, i.e., pointed, connected simplicial sets; $*$ is the basepoint or a one-point space. Then tow-$\mathscr{S}_0$ consists of towers in $\mathscr{S}_0$,

$$\cdots \to X_{s+1} \to X_s \to \cdots \to X_1 \to X_0 = *,$$

denoted $\{X_s\}$, and informally called a pro-space, with maps defined by

$$\mathrm{Hom}_{\text{tow-}\mathscr{S}_0}(\{X_s\}, \{Y_s\}) = \varprojlim_j \varinjlim_i \mathrm{Hom}_{\mathscr{S}_0}(X_i, Y_j).$$

Similar definitions apply to tow-$\mathscr{G}$ and tow-$\mathscr{A}$ where $\mathscr{G}$ is the category of groups, and $\mathscr{A}$ is the category of abelian groups.

For $n \geq 1$, the nth *homotopy pro-group* of $\{X_s\}$ is the pro-group $\{\pi_n X_s\}$. We say that two maps, f and g, from $\{X_s\}$ to $\{Y_s\}$ are *homotopic* if there is a map $h: \{X_s \times I\} \to \{Y_s\}$ such that the diagram

$$\begin{tikzcd} \{X_s \vee X_s\} \arrow[rr, "f \vee g"] \arrow[d] \arrow[drr] & & \{Y_s\} \\ \{X_s\} & & \{X_s \times I\} \arrow[ll] \arrow[u, "h"'] \end{tikzcd}$$

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