SOME COUNTABILITY CONDITIONS IN A COMMUTATIVE RING

BY

JIMMY T. ARNOLD, ROBERT GILMER,[1] AND WILLIAM HEINZER[2]

Introduction

Let R denote a commutative ring with identity and let $R[[X]]$ denote the ring of formal power series over R in an indeterminate X. If A is an ideal of R, then two naturally associated ideals of $R[[X]]$ are (1) $AR[[X]]$, the ideal of $R[[X]]$ generated by A, and (2) $A[[X]]$, the set of power series of $R[[X]]$ with coefficients all in A. It is clear that $AR[[X]]$ is contained in $A[[X]]$ and that if A is a finitely generated ideal, then equality holds. In general, however, $AR[[X]]$ may not be equal to $A[[X]]$, and it is noted in [19, p. 386] that the equality $AR[[X]] = A[[X]]$ holds precisely if the ideal A satisfies the following condition:

($*$) *If B is a countably generated ideal contained in A, then there exists a finitely generated ideal containing B and contained in A.*

We shall say that A is a ($*$)-*ideal* if the preceding condition is satisfied. It is clear that finitely generated ideals are ($*$)-ideals and that a countably generated ($*$)-ideal is finitely generated. An example in [19, p. 386, footnote 2] shows, however, that a ($*$)-ideal need not be finitely generated. If A is a ($*$)-ideal, then it can be shown that A satisfies the following condition (see Proposition 1.1).

($**$) *If $A_1 \subseteq A_2 \subseteq \cdots$ is an ascending sequence of ideals of R such that $\bigcup_{i=1}^{\infty} A_i = A$, then $A = A_i$ for some i.*

If A satisfies ($**$), then we call A a ($**$)-*ideal.* It is easy to see that R satisfies the ascending chain condition on ideals (that is, R is Noetherian) if and only if every ideal of R is a ($*$)-ideal if and only if every ideal of R is a ($**$)-ideal (see Proposition 1.2).

In analogy with a theorem of I. S. Cohen [10, Theorem 2] to the effect that R is Noetherian if each prime ideal of R is finitely generated, Arnold in [4, p. 20] asked if R is Noetherian, provided that each prime ideal is a ($*$)-ideal. Theorem 2.3 shows that the answer to Arnold's question is affirmative. We prove in Example 2.4 that the corresponding question for ($**$)-ideals has a negative answer—that is, there exist non-Noetherian rings in which each prime ideal is a ($**$)-ideal. Thus, in particular, there exist ($**$)-ideals that are not

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