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DEGREE OF BEST INVERSE APPROXIMATION BY POLYNOMIALS

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1. Introduction

Let π_n be the space of all algebraic polynomials with degree not exceeding n, and let $\|\cdot\|_p$ denote the L_p norm on the interval $[-1, 1]$. The purpose of this paper is the study of the speed of convergence of the error of *best inverse approximation* in L_p from π_n defined by

$$D_{n,p}(f) = \inf\{\|1 - fP_n\|_p \colon P_n \in \pi_n\}.$$

This problem finds its origin in the method of least-squares inverses which was introduced by E. A. Robinson (cf. [11] and [12, pages 153–175]), in 1963 in connection with deconvolution of 2-length wavelets and inverse filtering in geophysical studies. Since then, this method has also been adopted and generalized in recursive digital filter design (cf. [13]). The validity of these procedures and the mathematical theory have been discussed in [3], where the present problem was proposed.

To avoid trivial cases, we will always consider those functions f which are not identically zero, but with non-empty zero-sets in $[-1, 1]$ and $1 \le p < \infty$. The main result of this paper is the following.

THEOREM 1. *Let $f \not\equiv 0$ be a real analytic function on $[-1, 1]$ and $1 \le p < \infty$. There exist positive constants C_1, C_2 depending only on f with the following properties:*

(a) *if $f(x) = 0$ for some $x \in (-1, 1)$, then*

$$C_1 n^{-1/p} \le D_{n,p}(f) \le C_2 n^{-1/p}, \quad n = 1, 2, \ldots; \tag{1.1}$$

(b) *if $f(x) \neq 0$ for all $x \in (-1, 1)$ but $f(-1)f(1) = 0$, then*

$$C_1 n^{-2/p} \le D_{n,p}(f) \le C_2 n^{-2/p}, \quad n = 1, 2, \ldots. \tag{1.2}$$

It will be seen that the analyticity condition can be weakened. However, since our method of proof depends very heavily on the zero structure of an

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