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THE LATTICE OF GROUPS CONTAINING PSL(n,q) AND ACTING ON GRASSMANNIANS

BY

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Section 1

We consider here the set Ω of all subspaces of a fixed dimension inside a vector space. This set is technically called a Grassmannian. The special linear group has a natural representation on Ω, which we will show to be essentially maximal inside the symmetric group on Ω. More precisely, we have the following terminology and result.

Let V be an n-dimensional vector space over a finite field with q elements. Let $\Omega = \Omega(V,k)$ be the set of all k-dimensional subspaces of V. Then $P\Gamma L(n,q)$ has a faithful natural representation on $\Omega(n,k)$, which we will denote by $G_o = G_o(n,k)$. In the case $n = 2k$, (G_o,Ω) is permutation isomorphic to its dual, and we have natural graph automorphisms arising from the inverse transpose transformation. We define $\hat{G}_o = \langle G_o, j\rangle$ where j is any non-trivial graph automorphism of G_o. Observe that G_o has index 2 in $\hat{G}_o$, and all graph automorphisms are contained in $\hat{G}_o$. Let $S_o = S_o(n,k)$ be the representation of $PSL(n,q)$ on Ω. Denote by A_Ω the alternating group on Ω. Finally, let G be any subgroup of S_Ω containing S_o. We will prove:

THEOREM. *Suppose* $1 \le k \le n$ *and* $(n,k) \ne (2,1)$.

If $n \ne 2k$, *then* $G \subseteq G_o$ *or* $A_\Omega \subseteq G$.

If $n = 2k$, *then* $G \subseteq \hat{G}_o$ *or* $A_\Omega \subseteq G$.

There are questions concerning what occurs when we represent a Chevalley group on the cosets of a maximal parabolic subgroup. In particular, when is this group maximal in the alternating or symmetric group on these cosets? A maximal parabolic subgroup is maximal as a subgroup of its Chevalley group [9]. In the case of $PSL(n,q)$, the maximal parabolics fix k-dimensional subspaces for $1 \le k < n$. Therefore the representation of S_o on Ω is primitive. In our case, it's very easy to prove this directly. As the idea of the proof is used in a later lemma, we include it further on in our introduction.

The cases $k = 1$, $n \ge 3$ have already been solved by Kantor and McDonough [7]. Considering the dual space of V, the cases $k = n - 1$ with

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