ILLINOIS JOURNAL OF MATHEMATICS
Volume 28, Number 4, Winter 1984

ELEMENTARY MAPPINGS INTO IDEALS OF OPERATORS

BY

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1. Introduction

Let $\mathscr{H}$ be a separable, infinite dimensional complex Hilbert space, and let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$. For $n \geq 1$ and n-tuples of operators

$$A = (A_1, \ldots, A_n) \quad \text{and} \quad B = (B_1, \ldots, B_n),$$

let $R \equiv R(A, B)$ denote the *elementary operator* on $\mathscr{L}(\mathscr{H})$ defined by

$$R(X) = A_1XB_1 + \cdots + A_nXB_n \quad [18].$$

This prescription includes several special cases of interest, e.g., the inner derivations δ_A $(X \to AX - XA)$ [1], the left and right multiplications L_A and R_A $(X \to AX, X \to XA)$, the generalized derivations $T(A, B)$ $(X \to AX - XB)$ [9], and the elementary multiplication operators $S(A, B)$ $(X \to AXB$ [13].

In [11], C. K. Fong and A. R. Sourour described the case when Ran $(R(A, B))$, the range of the elementary operator $R(A, B)$, is contained in either the trivial ideal (0) or the ideal $\mathscr{K}(\mathscr{H})$ of all compact operators on $\mathscr{H}$. In considering the identity $R = 0$, Fong and Sourour reduce to the case when $\{B_1, \ldots, B_n\}$ is linearly independent, and show that in this case $R = 0$ if and only if $A_i = 0$ $(1 \leq i \leq n)$ [11, Theorem 1]. Analogously, they show that to study the inclusion Ran $(R(A, B)) \subset \mathscr{K}(\mathscr{H})$, it suffices to consider the case when $\{B_1, \ldots, B_n\}$ is independent modulo $\mathscr{K}(\mathscr{H})$; in this case, Ran $(R(A, B)) \subset \mathscr{K}(\mathscr{H})$ if and only if $A_i \in \mathscr{K}(\mathscr{H})$ $(1 \leq i \leq n)$ [11, Th. 3]. In [2], C. Apostol and L. Fialkow studied the problem of characterizing when the range of an elementary operator is contained in an arbitrary (two-sided) ideal of $\mathscr{L}(\mathscr{H})$. It is proved in [2, Theorem 1.1] that if $\{B_1, \ldots, B_n\}$ is independent modulo $\mathscr{K}(\mathscr{H})$ and $\mathscr{I}$ is a proper two-sided ideal of $\mathscr{L}(\mathscr{H})$, then Ran $(R(A, B)) \subset \mathscr{I}$ if and only if $A_i \in \mathscr{I}$ $(1 \leq i \leq n)$. (It is not difficult to see that the hypothesis of independence modulo $\mathscr{K}(\mathscr{H})$ cannot be weakened to independence modulo $\mathscr{I}$ [2].)

The range inclusion problem for elementary operators with arbitrary coefficient sequences remains unsolved, but in the sequel we take a first step

Received May 12, 1982.

[1] Research partially supported by a National Science Foundation grant.