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ON THE SIZES OF THE SETS OF INVARIANT MEANS

BY

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1. Introduction and notation

Let G be a locally compact group with a fixed Haar measure λ. If G is compact, we assume $\lambda(G) = 1$. Let $L^p(G)$ be the associated real Lebesgue spaces ($1 \leq p \leq \infty$). For each $f \in L^\infty(G)$ and $x \in G$, ${}_x f \in L^\infty(G)$ is defined by ${}_x f(y) = f(xy)$, $y \in G$. Let P denote the set of all $f \in L^1(G)$ with $f \geq 0$ and $\|f\|_1 = \int_G |f(x)|\, dx = 1$. A functional $m \in L^\infty(G)^*$ is called a mean if $m(1) = 1$ and $m(f) \geq 0$ for each $f \in L^\infty(G)$ with $f \geq 0$. We denote the set of all left invariant means on $L^\infty(G)$ by LIM, i.e. all the mean m with $m({}_x f) = m(f)$, ($x \in G$, $f \in L^\infty(G)$). For $\varphi \in P$ and $f \in L^\infty(G)$, $\varphi * f \in L^\infty(G)$ is defined by

$$\varphi * f(x) = \int_G \varphi(t) f(t^{-1}x)\, dt, \quad x \in G,$$

and the set of all topologically left invariant means, i.e. the mean m on $L^\infty(G)$ with $m(\varphi * f) = m(f)$ ($\varphi \in P$, $f \in L^\infty(G)$), is denoted by $TLIM$. For any set A, the cardinality of A is denoted by $|A|$.

Let $CB(G)$ be the Banach space of continuous bounded functions on G in the supremum norm. We can define a left invariant mean on $CB(G)$ as in the case of $L^\infty(G)$. We denote all left invariant means and all topologically left invariant means on $CB(G)$ by $LIM(CB(G))$ and $TLIM(CB(G))$, respectively. When $LIM \neq \phi$, we say that G is amenable. It is well known that any topologically left invariant mean is left invariant and G is amenable if and only if one of the following conditions is true: (a) $TLIM \neq \phi$. (b) $LIM(CB(G)) \neq \phi$. (c) $TLIM(CB(G)) \neq \phi$. Also, if G is amenable as a discrete group, then G is amenable (see [9]).

The size of $LIM \sim TLIM$ was first studied by Granirer [7] and Rudin [18]. They showed independently that $LIM \sim TLIM \neq \phi$ if G is nondiscrete and

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