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SYMMETRIC CLOSED OPERATORS COMMUTING WITH A UNITARY TYPE I REPRESENTATION OF FINITE MULTIPLICITY ARE SELF-ADJOINT

BY

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Introduction

In the first part of this paper we prove the following:

THEOREM. *Let τ be a type I unitary representation of a group G whose direct integral decomposition has (a.e.) finite multiplicity. Let $(T, \mathscr{D}_T)$ be a densely defined symmetric operator such that $\tau(x)\mathscr{D}_T = \mathscr{D}_T$ and $\tau(x)Tf = T\tau(x)f$ for all $x \in G$ and $f \in \mathscr{D}_T$. Then T is essentially self-adjoint.*

In [1] van der Ban proves a similar result about selfadjointness for symmetric spaces of semisimple groups "filling some gaps in the argument of [4]" and proves the finiteness of multiplicities in the corresponding representation and on more general symmetric spaces. The two parts of the paper are independent; thus our result can be used with the second part of [1] to get the main theorem in the first part of [1].

For another application of the theorem see [2], where invariant differential operators occur in connection with nilpotent groups and induced representations.

In the second part of the paper we analyse commutative algebras of unbounded invariant operators such as occur in the papers [1] and [2].

In [7] we give an example of a left invariant symmetric differential operator on the Heisenberg group which fails to be essentially selfadjoint. (If X, Y, Z is the usual basis of the Lie algebra the operator $X^4 + Y^2$ is such an example). This shows that even in the case of an invariant symmetric differential operator on a homogeneous space the assumption of finite multiplicity in the above theorem is essential.

If τ is multiplicity-free (so that the commutant of τ is abelian) the above theorem is easy to prove: Since the closure of T is invariant we may assume

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