NONCOMMUTATIVE BANACH ALGEBRAS AND ALMOST PERIODIC FUNCTIONS[1]

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1. Introduction

A structure theory is developed for a class of Banach algebras which we call inner product algebras (IP-algebras). We were led to these algebras by the algebra of almost periodic functions under convolution.

Let $A = \mathrm{AP}(G)$ be the set of all almost periodic functions on a topological group G considered as a Banach algebra under the norm $\| f \| = \sup | f(t) |$, pointwise addition, and convolution multiplication. This algebra is rich in structure. Not only is it a Banach algebra in the norm $\| f \|$, but also it is a pre-Hilbert space in the norm $| f | = (f, f)^{1/2}$, where the inner product is given by $(f, g) = M_t[f(t)\overline{g(t)}]$ (here M is the mean-value functional of von Neumann [8]). This pre-Hilbert space is, in general, not complete (even for G the real numbers). Denote the convolution of f and g by fg where $fg(s) = M_t[f(st^{-1})g(t)]$ [8, p. 456]. The two norms are connected [7], [8] by (1) $| f | \leqq \| f \|$ and (2) $\| fg \| \leqq | f | \, | g |$ for all $f, g \in A$. Also (3) $Af = 0$ implies $f = 0$. Moreover the natural involution $f \rightarrow f^*$ defined by $f^*(t) = \overline{f(t^{-1})}$ satisfies (4) $(fg, h) = (g, f^*h) = (f, hg^*)$ for all $f, g, h \in A$. Also (5) f lies in the closure of fA for each $f \in A$ [8, Theorem 17]. Our interest in $\mathrm{AP}(G)$ from the point of view of the general theory of Banach algebras began with the discovery that any Banach algebra with an involution which is a pre-Hilbert space satisfying conditions (1)–(5) (or even weaker conditions, see Theorem 4.9) is a semisimple dual Banach algebra.

A somewhat analogous situation was treated by Ambrose [1] who started with the L_2-algebra of a compact group as a model and abstracted to H^*-algebras. Likewise starting with $\mathrm{AP}(G)$ we abstract to what we call IP-algebras and right IP-algebras.[2] As in [1] our main goal is a structure theory for the algebras under consideration. We have, at the same time, been able to manage with requirements substantially weaker than those numbered above.

Let A be a Banach algebra which is also a pre-Hilbert space (A_h) in terms of the norm $| f |$. Suppose that, as in (1) and (3), convergence in the norm $\| f \|$ implies convergence in $| f |$ and $Af = 0$ implies $f = 0$. We call A a right IP-algebra if there exists a dense right ideal $\mathfrak{B}_r$ such that each $f \in \mathfrak{B}_r$ has a

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[2] Actually we consider an analogue of the right H^*-algebras of Smiley [13] as well.