ON A THEOREM OF CHERN

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This paper tells an old story in a more modern setting. One of the more striking results in global differential geometry is the theorem of Chern (Theorem 11 in [1]) relating the characteristic classes of a vector bundle to the curvature of this bundle. A special case of this result is the Gauss-Bonnet theorem for almost complex manifolds.

There are two approaches to characteristic classes which will be discussed here. The first is by using the obstruction theory of fibre bundles as in [1] and [2]. It is here that the differential geometry of the fibre bundle enters essentially, and, although there is a very simple underlying principle, the computations tend to be lengthy and obscure the geometry. Our approach has been first to utilize the geometric principle for simple types of bundles (Proposition 2), where it is hoped that the computations have significant geometric content. Having done this, we pass to more complicated bundles by algebraic means; the geometric interpretation remains the same. The second approach is that introduced by Kodaira and Spencer in applying the theory of Chern classes to algebraic geometry. Again, we try to illustrate in a simple but meaningful way how the curvature enters (Proposition 3) and is useful for the theory of complex manifolds. Again we pass from this result to more complicated bundles by using what might be called the "induction principle in fibre bundle theory" due to Chern [2].

This paper is expository in nature, and the expert will find little that is new. In particular, we have reproduced a proof (Proposition 5) which seems to exist only in lecture notes. The point of view adopted is differential-geometric, and we have freely used results here (reference [4]). On the other hand, we have tried to utilize only general techniques in topology, avoiding specific results. As a single exception, we have used some homology properties of the Grassmann variety in §§4 and 5.

1. Preliminaries

We shall work within the framework of compact almost complex manifolds (abbreviated a.c. manifolds); if X is one such, the a.c. structure tensor is denoted by J_X. An almost complex mapping is one whose differential commutes with the a.c. structure tensors. We define an a.c. principal bundle $GL(r, \mathbf{C}) \to P_\xi \xrightarrow{\pi} X$ (where $GL(r, \mathbf{C}) \to P_\xi$ is injection of the fibre, and π is the bundle projection) such that (i) J_{P_ξ} is invariant under $GL(r,\mathbf{C})$ acting on the right, (ii) J_{P_ξ} restricted to a fibre gives the a.c. structure of the fibre,

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