THE RADIUS OF UNIVALENCE OF CERTAIN ENTIRE FUNCTIONS

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It was shown in [1] (see also [5]) that the radius of univalence, $R_U(\nu)$, of the function $z^{1-\nu}J_\nu(z)$, where $J_\nu(z)$ is the usual Bessel function $(\nu > 0)$, is the smallest positive zero of its derivative, and two-sided inequalities were obtained for $R_U(\nu)$. In this note we give a short proof of a more general result, which delineates a rather broad class of entire functions for which the same conclusion holds. Further, we refine the inequalities mentioned above to sharper ones which give asymptotic equalities for $\nu \to \infty$. The basic idea is simply that whereas the radius of univalence is quite troublesome to deal with directly, the radius of starlikeness is obtainable almost immediately from Hadamard's factorization.

Let $\mathfrak{F}$ be a Montel compact [2] family of functions

$$f(z) = z + a_2 z^2 + \cdots, \tag{1}$$

regular in $|z| < 1$, and put $\gamma_n = \max_{f\epsilon\mathfrak{F}} |a_n|$ $(n = 2, 3, \cdots)$. If

$$g(z) = z + b_2 z^2 + \cdots \tag{2}$$

is a given entire function, then the $\mathfrak{F}$-radius, $R_{\mathfrak{F}}$, of $g(z)$ is

$$R_{\mathfrak{F}} = \sup \{R \mid R^{-1}g(Rz) \in \mathfrak{F}\}. \tag{3}$$

The inequalities $|b_n|R^{n-1} \leqq \gamma_n$ $(n = 2, 3, \cdots)$ which must hold for all $R \leqq R_{\mathfrak{F}}$, show first that either $R_{\mathfrak{F}} < \infty$ or $g(z) \equiv z$, and second that

$$R_{\mathfrak{F}} \leqq \min_{n\leqq 2} \{\gamma_n / |b_n|\}^{1/(n-1)} \tag{4}$$

We consider the families (T) of typically real functions, (U) of univalent functions, (S) of starlike univalent functions, and (C) of convex univalent functions. If $g(z)$ in (2) has real coefficients, then plainly

$$R_C \leqq R_S \leqq R_U \leqq R_T \tag{5}$$

since a univalent function with real coefficients is typically real.

Now let G denote the class of entire functions of either of the following two forms:

$$(6)\quad \begin{aligned} &\text{(a)} \quad g(z) = ze^{\beta z}\textstyle\prod_{n=1}^{\infty} (1 + z/a_n), \\ &\text{(b)} \quad \beta \geqq 0; \quad 0 < a_1 \leqq a_2 \leqq \cdots; \quad \textstyle\sum |a_n|^{-1} < \infty, \end{aligned}$$

or

$$(7)\quad \begin{aligned} &\text{(a)} \quad g(z) = z\textstyle\prod_{n=1}^{\infty} (1 - z^2/a_n^2), \\ &\text{(b)} \quad 0 < a_1 \leqq a_2 \leqq \cdots; \quad \textstyle\sum |a_n|^{-2} < \infty. \end{aligned}$$

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