ON COMPLEMENTARY AUTOMORPHIC FORMS AND SUPPLEMENTARY FOURIER SERIES

Dedicated to Hans Rademacher on the occasion of his seventieth birthday

BY

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1. Let Γ be a discontinuous group of linear transformations of the upper half-plane $\mathcal{H}$ on itself. If F, G are automorphic forms belonging to Γ, we say that F and G are *complementary forms* provided FG is a differential, i.e., provided

$$FG \in \{\Gamma, -2, 1\},$$

where by $\{\Gamma, -r, v\}$ we mean the complex vector space of automorphic forms of dimension $-r$ belonging to Γ and the multiplier system v. If $F \in \{\Gamma, -r, v\}$, $G \in \{\Gamma, -r', v'\}$, then F is complementary to G if and only if[2]

$$r + r' = 2, \qquad vv' = 1. \tag{1}$$

In particular, assume $r < 0$, μ a positive integer, and let $F_\mu(\tau)$ be that form in $\{\Gamma, -r, v\}$ which is regular in $\mathcal{H}$, has a pole of order μ at ∞, and has the Fourier expansion

$$e(-\kappa\tau/\lambda)F_\mu(\tau) = t^{-\mu} + \textstyle\sum_{m=0}^{\infty} a_m t^m, \qquad t = e(\tau/\lambda), \tag{2}$$

where

$$e(z) = e^{2\pi i z}.$$

Here λ, κ are defined in (3), (4). That is, we assume such a form exists. Petersson has defined a system of forms belonging to the complementary class $\{\Gamma, -r', v'\}$, namely, the Poincaré series $G(\tau, -r', v', \mu) = G_\mu$; cf. (8). In §3, Theorem 1, we shall exhibit a connection between F_μ and G: *If there exists a form $F_\mu \in \{\Gamma, -r, v\}$ satisfying* (2), *then $G_{\mu-1} \equiv 0$ when $\kappa > 0$ and $G_\mu \equiv 0$ when $\kappa = 0$.* This result is applied to the modular group in §4 (cf. Petersson [3, p. 432]).

If $F_\mu \in \{\Gamma, -r, v\}$, the coefficients a_m have convergent series representations given in (5). But a_m can be defined by (5) *whether F_μ is an automorphic form or not.* Write

$$a_m = a_m(\mu, -r, v)$$

to express the fact that a_m is determined by the data in parentheses even though there may not exist a form of type F_μ.

In §5 we restrict $-r$ to positive integral values. We regard F_μ as a Fourier

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[2] It is known that v^{-1} is a multiplier system for $[\Gamma, -r']$ if v is one for $[\Gamma, -r]$.