ON THE NUMBER OF MATRICES WITH GIVEN CHARACTERISTIC POLYNOMIAL

BY

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1. Introduction

Let K be a finite field with q elements, and let K_n denote the ring of all $n \times n$ matrices with entries in K. Recently Fine and Herstein proved[2]

The number of nilpotent matrices in K_n is q^{n^2-n}.

We shall prove here the following generalizations.

THEOREM 1. *Let $f(x)$ be an irreducible polynomial in $K[x]$ of degree $d \geqq 1$. Then the number of matrices $X \in K_{nd}$ for which $f(X)$ is nilpotent is*

$$(1) \qquad q^{n^2d^2-nd} \cdot \frac{(1-q^{-1})(1-q^{-2}) \cdots (1-q^{-nd})}{(1-q^{-d})(1-q^{-2d}) \cdots (1-q^{-nd})}.$$

Before stating the second result to be proved here, which includes the above theorem as a special case, we introduce some notation. Define

$$(2) \qquad F(u, r) = (1-u^{-1})(1-u^{-2}) \cdots (1-u^{-r}),$$

where $F(u, 0) = 1$. Then we have[3]

THEOREM 2. *Let $g(x) \in K[x]$ be a polynomial of degree n, and let*

$$(3) \qquad g(x) = f_1^{n_1}(x) \cdots f_k^{n_k}(x)$$

be its factorization in $K[x]$ into powers of distinct irreducible polynomials $f_1(x), \cdots, f_k(x)$. Set

$$d_i = \text{degree of } f_i(x), \qquad 1 \leqq i \leqq k.$$

Then the number of matrices $X \in K_n$ with characteristic polynomial $g(x)$ is

$$(4) \qquad q^{n^2-n} \cdot \frac{F(q, n)}{\prod_{i=1}^{k} F(q^{d_i}, n_i)}.$$

The proofs of these theorems do not require a knowledge of the Fine-Herstein paper, except for the following combinatorial lemma which they establish and which we state without proof.

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[2] N. J. FINE AND I. N. HERSTEIN, *The probability that a matrix be nilpotent*, Illinois J. Math., vol. 2 (1958), pp. 499–504.

[3] Another proof of Theorems 1 and 2 is given by M. GERSTENHABER, *On the number of nilpotent matrices with coefficients in a finite field*, Illinois J. Math., vol. 5 (1961), pp. 330–333.