A CONDITION FOR THE SOLVABILITY OF A FINITE GROUP

BY

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In [8] H. Wielandt introduced the concept of subinvariant subgroup and proved that the set of all subinvariant subgroups of a finite group G form a lattice, $\mathcal{S}(G)$, under the usual compositions of intersection and subgroup union. (Cf. [2], Chapter 8.) Clearly the definition of solvability for G requires $\mathcal{S}(G)$ to form a rather substantial skeleton for $\mathcal{L}(G)$, the lattice of all subgroups of G under the same compositions, and suggests that there exist relations between $\mathcal{S}(G)$ and $\mathcal{L}(G)$ which insure the solvability of G. One such relationship was given by Wielandt in [8]: A finite group G is nilpotent if and only if $\mathcal{L}(G) = \mathcal{S}(G)$.

Now a direct extension of a portion of this result based only on the ratio of the number of elements in $\mathcal{L}(G)$ to the number in $\mathcal{S}(G)$ is impossible as the direct product of a simple nonabelian group of small order with a nilpotent group of large order indicates. Thus the distribution of the elements of $\mathcal{S}(G)$ in $\mathcal{L}(G)$ must be considered. We prove here that, stated roughly, if the elements of $\mathcal{S}(G)$ comprise over 20% of $\mathcal{L}(G)$ and are rather uniformly distributed throughout $\mathcal{L}(G)$, then G is a solvable group.

1. On maximal subgroups

Two intermediate results essential to the proof of the theorem mentioned above are proved in this section. Both are results concerning maximal subgroups and are of some interest in themselves.

Theorem 1. *If the finite group G contains a maximal subgroup M which is nilpotent of class less than* 3, *then G is solvable.*

This result is properly contained in a theorem of B. Huppert [4] *except* when M contains a 2-subgroup of class 2. However this rather special case attains a degree of importance from some work of N. Itô and of J. G. Thompson. In [6] Thompson announced that a finite group which contains as a maximal subgroup a nilpotent group of odd order is a solvable group, while in [5] Itô showed that certain nonsolvable linear fractional groups contain maximal subgroups which are nilpotent (and of even order).

Now if the theorem is not true, then among the nonsolvable groups which contain as maximal subgroups nilpotent groups of class less than 3 there is one (at least) of minimal order. Denote such a group by G. We shall show that G cannot exist and thereby prove the theorem.

First of all G contains no normal subgroups $(\neq(1))$ which lie entirely in M. For if $K \subseteq M$ is normal in G, then G/K satisfies the hypothesis of the theorem. Since the order of G/K, $o(G/K)$, is less than $o(G)$, G/K is solv-

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