HAUSDORFF DIMENSION IN PROBABILITY THEORY II

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1. Introduction and definitions

Let $\{x_1, x_2, \cdots\}$ be a stochastic process, with finite or countable state space σ, defined on a probability measure space $(\Omega, \mathcal{B}, \mu)$. In [1] a Hausdorff dimension $\dim_\mu M$ was defined for each set $M \subset \Omega$, in the following way.[1] A cylinder (of rank n) is defined to be a set of the form $\{\omega: x_k(\omega) = a_k, k = 1, 2, \cdots, n\}$, where $a_k \in \sigma$. If $M \subset \Omega$ and $\rho > 0$, a μ-ρ-covering of M is a finite or countable collection $\{v_i\}$ of cylinders such that $M \subset \bigcup_i v_i$ and $\mu(v_i) < \rho$ for each i. If $\rho, \alpha > 0$, put $L_\mu(M, \alpha, \rho) = \inf \sum_i \mu(v_i)^\alpha$, where the infimum extends over all μ-ρ-coverings $\{v_i\}$ of M, and let $L_\mu(M, \alpha) = \lim_{\rho \to 0} L_\mu(M, \alpha, \rho)$. If $L_\mu(M, \alpha) < \infty$, then $L_\mu(M, \alpha + \varepsilon) = 0$ for all $\varepsilon > 0$; hence we can define

$$\dim_\mu M = \sup\{\alpha: L_\mu(M, \alpha) = \infty\} = \inf\{\alpha: L_\mu(M, \alpha) = 0\}. \tag{1.1}$$

It was shown in [1] that if Ω is the unit interval $(0, 1]$, if μ is Lebesgue measure, and if $\sum_{n=1}^\infty x_n(\omega) s^{-n}$ is, for each ω, the nonterminating base s expansion of ω, then this definition reduces to the classical one due to Hausdorff.

The dimension of M depends both on the measure μ and the process $\{x_n\}$. The dependence upon $\{x_n\}$ is not exhibited in the notation $\dim_\mu M$, since $\{x_n\}$ will remain fixed throughout the discussion. However, we will consider several measures μ simultaneously, and the main purpose of the paper is to investigate how $\dim_\mu M$ varies as μ varies. For $\omega \in \Omega$ and $n = 1, 2, \cdots$, put

$$u_n(\omega) = \{\omega': x_k(\omega') = x_k(\omega), k = 1, 2, \cdots, n\}.$$

In other words, $u_n(\omega)$ is that cylinder of rank n which contains ω. In §2 we prove several refinements of the fact that if μ and ν are probability measures on $\mathcal{B}$, and if

$$M \subset \left\{\omega: \lim_{n \to \infty} \frac{\lg \nu(u_n(\omega))}{\lg \mu(u_n(\omega))} = \delta\right\}, \tag{1.2}$$

then

$$\dim_\mu M = \delta \dim_\nu M. \tag{1.3}$$

In §3, the results of §2 are used to extend and simplify some of the theorems of [1]. The essential idea here is to compute $\dim_\mu M$ for certain sets M by constructing a measure ν such that (1.2) holds and such that $\dim_\nu M = 1$. It then follows from (1.3) that $\dim_\mu M = \delta$. Finally, §4 contains some re-

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[1] In [1] the state space σ was assumed to be finite, but the definition applies to a countable σ as well.