GROUPS WITH REPRESENTATIONS OF BOUNDED DEGREE II

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Kaplansky has initiated in [3] a study of infinite groups G all of whose irreducible representations are of bounded degree. It was shown in [3] that if G contains a normal abelian subgroup of finite index, then all the irreducible representations of G are of bounded degree, and a bound was obtained utilizing identities of finite matrix rings and the theory of Banach algebras.

With the additional information we have about identities of matrix rings and of discrete group algebras, we are able to obtain more concrete results in this direction. In particular we obtain among others the result that if G contains any abelian (not necessarily normal) subgroup of index n in G, then all representations of G are of degree $\leqq n$.

The converse is not true even for $n = 2$. In this case we determine all groups whose irreducible representation are finite-dimensional of degree $\leqq 2$, and show that they belong to two types: (1) groups G having a normal abelian subgroup of index 2; (2) groups G having a center N such that G/N is an abelian 2-group of order 8.

1. Finite-dimensional representations

In what follows all representations are considered over fields of characteristic zero.

The following notations and results will be used:

$$[x_1, x_2, \cdots, x_k] = \sum \pm x_{i_1} x_{i_2} \cdots x_{i_k},$$

where x_i are noncommutative indeterminates and the sum ranges over all permutations $(i_1, \cdots, i_k)$ of the first k letters and the sign is positive for even permutations and negative for odd permutations. It is well known [4] that matrix rings F_n satisfy the identity $[x_1, x_2, \cdots, x_{2n}] = 0$ and no identities of lower degree.

Following [3], a group G is said to have the property P_k if for any k elements $g_1, \cdots, g_k$ of G, the $k!/2$ products $g_{i_1} g_{i_2} \cdots g_{i_k}$ obtained from all even permutations is identical with the $k!/2$ products obtained from the odd permutations. The fact that G has property P_k is equivalent to the group ring $F[G]$ (for arbitrary field of characteristic zero) satisfying the identity

$$[x_1, x_2, \cdots, x_k] = 0.$$

Let V be a finite-dimensional vector space over some field C, and let it be also a representation space of G. V determines an *absolutely irreducible representation* of G, if $V \otimes_C F$ is G-irreducible for all field extensions F of C.

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