AVERAGE ORDER OF ARITHMETIC FUNCTIONS

BY

J. P. TULL

1. Introduction and an elementary lemma

The author has given a theorem [8] by which it is possible to find an asymptotic formula for the summatory function of the convolution of two arithmetic functions if such a formula is known for these functions. By the convolution of arithmetic functions a and b we mean

$$(a * b)(n) = \sum_{d \mid n} a(d)\, b(n/d).$$

If $A(x) = \sum_{n<x} a(n)$ and $B(x) = \sum_{n<x} b(n)$, we have used the term *Stieltjes resultant* for the function

$$C(x) = \sum_{n<x} (a * b)(n)$$

due to the fact that for almost all x

$$C(x) = \int_1^x A(x/u)\, dB(u).$$

However, the term *convolution* is just as natural, and so we have two convolutions, $*$ and $\times$, where for $x \geqq 1$

$$(A \times B)(x) = \sum_{n<x} (a * b)(n).$$

In the present paper we shall apply the theorem of [8] to some interesting arithmetic functions and then apply the following elementary lemma to some of these results and also to some known nonelementary asymptotic formulae to find estimates for sums $\sum_{n<x} a(n)/n$.

LEMMA. *Given an arithmetic function a, if for $x \geqq 1$*

$$A(x) = \sum_{n<x} a(n) = R(x) + O(x^{\alpha} L(x)),$$

where R is continuous on $[1, \infty)$, *α is real, L slowly oscillating (see below), then*

$$\sum_{n<x} a(n)/n = \int_1^x R(t)t^{-2}\, dt + R(x)x^{-1} + c + O(x^{\alpha-1} L_1(x)),$$

where $c = 0$ if $\alpha \geqq 1$,

$$c = \int_1^{\infty} t^{-2}(A(t) - R(t))\, dt$$

if $\alpha < 1$, $L_1(x) = L(x)$ if $\alpha \neq 1$, and

$$L_1(x) = \int_1^x t^{-1} L(t)\, dt$$

if $\alpha = 1$.

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