THE COHOMOLOGY THEORY OF A PAIR OF GROUPS[1]

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1. Introduction

In a series of papers by S. Eilenberg and S. Mac Lane [4], [5] and by S. Mac Lane [12], the cohomology theory of groups has been expounded in such a way that the group extension problem is recast in homological terms. (See also [9].) In particular, those authors were able to show that group extensions can be related to appropriate 2-cohomology classes in the abelian case, while in the non-abelian case the possibility of extension depends upon a certain obstruction, a 3-cocycle, becoming a coboundary. Let us suppose that we are given an abelian group A with two groups of operators, B_1 and B_2, where each operator from B_2 commutes with each operator from B_1. As in R. Baer [2], one can set up cochains, cocycles, coboundaries, and cohomology classes (herein referred to with the prefix *bi*, as in *bicocycle*) for this pair of groups B_1, B_2 with coefficients in A. In §2, using resolutions, we show that the various bicohomology groups $\mathfrak{H}^{(n)}(B_1, B_2; A)$ of the pair B_1, B_2 are isomorphic to the corresponding cohomology groups of the direct sum $B = B_1 \oplus B_2$. In fact, we can find a specific map $\mathfrak{F}$ over the identity automorphism on the group of integers Z from the tensor product of the standard projective resolutions of Z as a left $Z(B_1)$-module and as a left $Z(B_2)$-module to the standard projective resolution of Z as a left $Z(B)$-module. In §3, we consider an extension G of A by B, letting ω be the corresponding epimorphism from G to B. Then the subgroups $G_k = \omega^{-1}B_k$ extend A by B_l $(l \neq k)$ and have the property that each operator b_k (from B_k) on A extends to an automorphism of G_l which induces the identity automorphism on $G_l/A \cong B_l$ so that, as elements in A, (where $u_k(b_k)$ represents b_k in G_k),

$$[u_1(b_1)]^{-1}b_2[u_1(b_1)] + [u_2(b_2)]^{-1}b_1[u_2(b_2)] = 0.$$

Such pairs of extensions, G_1, G_2 of A by B_1, B_2, are called *coherent*. Conversely, given such a pair of coherent extensions, we can find, using the map $\mathfrak{F}$, an extension G of A by B with epimorphism ω from G to B such that each $G_k = \omega^{-1}B_k$. The set of coherent pairs of extensions of A by B_1, B_2 can be made into a group $\mathfrak{T}(B_1, B_2; A)$ which is an epimorphic image of $\mathfrak{H}^{(2)}(B_1, B_2; A)$ where the kernel is the inverse image of the coherent pair of splitting extensions. We map both $\mathfrak{H}^{(2)}$ and $\mathfrak{T}$ above into $\mathfrak{H}^{(2)}(B_1, A) \oplus \mathfrak{H}^{(2)}(B_2, A)$, forming part of an exact diagram.

In §4, we show that the group of autoequivalences of G over A by B (the

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